Ann. For. Res. 66(1): 4394-4402, 2023
ISSN: 18448135, 20652445
ANNALS OF FOREST RESEARCH
www.e-afr.org

AN INVESTIGATION OF THE USE OF ORDINARY DIFFERENTIAL EQUATIONS APPLIED TO MECHANICAL ENGINEERING PROBLEMS.

Luis Fernando Buenaño Moyano [a] *, Celin Abad Padilla Padilla [a], Diego Armando Tacle Humanante [b], Fabián Eduardo Vilema Chuiza [b], Diego Javier Álvarez Lara [c].
[a]Escuela Superior Politécnica de Chimborazo (ESPOCH), Riobamba, 060155, Ecuador
[b]Instituto Superior Tecnológico Carlos Cisneros, Riobamba, 060155, Ecuador
[c]Investigador Independinete.

Abstract
Ordinary Differential Equations (ODEs) are fundamental mathematical tools for modeling and analyzing mechanical systems in engineering. ODEs are used to describe the motion of physical systems, such as the motion of a mass on a spring, the behavior of a vibrating structure, or the dynamics of a mechanical system subjected to external forces. This paper presents a review of the applications of ODEs in mechanical engineering, highlighting their importance in modeling and simulating various mechanical systems. We discuss different types of ODEs and the methods used to solve them. In addition, we present several examples of ODE-based models of mechanical systems, including problems related to robotics, vehicle dynamics, and structural analysis.
Keywords: Differential Equations; mechanical problems; Laplace transform; equations of motion; Runge-Kutta method; mechanical problems; equations of motion; Spring-mass systems; Phyton.

Introduction

Mechanical engineering is concerned with the design, analysis, and manufacturing of machines and mechanical systems. To understand and design such systems, it is essential to have a thorough understanding of the underlying physics and mathematics. One of the most powerful tools in this regard is ordinary differential equations (ODEs). ODEs allow us to model the behavior of mechanical systems mathematically, and the solutions of these equations give us insight into the system's response to different inputs and conditions [1,2]. ODEs are also used to develop control systems that help to optimize the performance of mechanical systems.

Ordinary Differential Equations (ODEs) play a fundamental role in modeling and solving problems in mechanical engineering. In many cases, the behavior of mechanical systems can be described by a set of differential equations, which relate the forces and motions of the system [3–5]. Solving these equations enables engineers to predict the behavior of a system under different conditions and to design systems that meet specific performance requirements.

ODEs can be used to model a wide range of mechanical engineering problems, such as the motion of vehicles, aircraft, and spacecraft, the behavior of mechanical systems such as robots, machines, and engines, and the vibrations of structures and materials [6,7]. They are also essential for designing control systems that regulate the behavior of mechanical systems.

Mechanical systems are ubiquitous in engineering, from vehicles and machines to buildings and bridges. The behavior of these systems can be complex and difficult to understand, making mathematical modeling a critical tool for design and analysis [1,8,9]. Ordinary Differential Equations (ODEs) are one of the most powerful tools for modeling and simulating mechanical systems, providing engineers with the ability to predict system behavior and optimize designs.

ODEs describe the behavior of a system in terms of its motion or evolution over time. In mechanical engineering, ODEs are used to describe the motion of a mass on a spring, the behavior of a vibrating structure, or the dynamics of a mechanical system subjected to external forces [10–14]. ODEs can be linear or nonlinear and can have constant or time-varying coefficients. Solving ODEs can be challenging, but a variety of numerical methods are available to simulate the behavior of complex mechanical systems [15,16].

Types of ODEs:

There are several types of ODEs that are commonly used in mechanical engineering, including first-order ODEs, second-order ODEs, and higher-order ODEs [7,17]. First-order ODEs describe the evolution of a single variable, such as the position of a mass, and can be written as:

$$\frac{dy}{dt} = f(t, y)$$

where y is the dependent variable, t is the independent variable, and f(t, y) is a function of both t and y. Second-order ODEs describe the evolution of two variables, such as the position and velocity of a mass, and can be written as:

$$\frac{d^2y}{dt^2} = f(t, y, dy/dt)$$

Higher-order ODEs describe the evolution of more than two variables and can be reduced to a system of first-order ODEs.

Methods for solving ODEs:

ODEs and Numerical Methods: An ordinary differential equation is an equation that relates a function and its derivatives with respect to a single independent variable. The general form of an ODE is: $y'=f(x,y)$ where y' represents the derivative of y with respect to x, and f(x,y) is a function of x and y. The solution of an ODE involves finding a function y(x) that satisfies the given differential equation. Analytical solutions are not always possible, and numerical methods are used to solve ODEs in such cases [13,18–20]. The most commonly used numerical methods for solving ODEs are the Runge-Kutta and the Euler methods.

There are several methods for solving ODEs, including analytical methods and numerical methods. Analytical methods involve finding an explicit solution to the ODE, while numerical methods involve approximating the solution using a sequence of discrete time steps. Analytical methods are often limited to simple ODEs with known solutions, while numerical methods can handle more complex ODEs and provide approximate solutions.

Numerical methods for solving ODEs include the Euler method, the Runge-Kutta method, and the Adams-Bashforth method. These methods involve approximating the solution using a sequence of discrete time steps, with the accuracy of the approximation increasing as the time step size decreases. Numerical methods can be computationally intensive, but they provide a powerful tool for simulating the behavior of complex mechanical systems.

Applications of ODEs in mechanical engineering:

ODEs have many applications in mechanical engineering, ranging from the design of robotic systems to the analysis of vehicle dynamics and structural analysis. In robotics, ODEs are used to model the motion and behavior of robotic systems, including their interaction with the environment. In vehicle dynamics, ODEs are used to model the motion and behavior of vehicles, including their response to steering and braking inputs. In structural analysis, ODEs are used to model the behavior [12,21,22].

ODEs in Modeling Mechanical Systems:

ODEs are used to model a wide range of mechanical systems, from simple spring-mass-damper systems to more complex systems such as engines, turbines, and pumps. In a spring-mass-damper system, the displacement of a mass attached to a spring and damper is governed by a second-order ODE. This system is used to model a wide range of mechanical systems, from simple harmonic oscillators to more complex systems such as vehicle suspension systems. Other examples of mechanical systems modeled using ODEs include fluid mechanics, heat transfer, and vibration analysis.

ODEs in Vibration Analysis:

Vibration analysis is an essential area of mechanical engineering that involves the study of the response of mechanical systems to external forces. Vibration analysis involves the solution of ODEs, which describe the motion of a vibrating system. The study of vibrations is critical in the design and development of machines and mechanical systems, as it helps to identify potential failure modes and ways to mitigate them.

ODEs in Heat Transfer:

Heat transfer is another area of mechanical engineering where ODEs are widely used. The heat transfer equation, which is a partial differential equation, is often simplified using ODEs to model heat transfer in different types of systems. ODEs are used to model the temperature distribution in materials and systems, such as heat exchangers and thermal insulation.

ODEs in Fluid Mechanics:

Fluid mechanics is an area of mechanical engineering that deals with the study of fluids and their behavior in different systems. ODEs are used to model fluid flow in pipes, pumps, and turbines,

as well as the dynamics of fluid-structure interactions. These models are used to design and optimize different types of fluid systems, including pipelines, pumps, and engines.
ODEs can be solved using a variety of numerical methods, including finite difference methods, finite element methods, and numerical integration. These methods allow engineers to obtain accurate solutions to complex systems that cannot be solved analytically. The solutions obtained from ODEs can be used to optimize designs, predict system behavior under various conditions, and develop control systems that ensure stable and efficient operation.

Applications

Exercise 1. Spring-mass-damper system

Let's consider a simple spring-mass-damper system, which is a common example in classical mechanics. In this system, a mass is attached to a spring, which in turn is attached to a fixed point. The motion of the mass is damped by a frictional force proportional to the velocity of the mass.
The equation of motion for this system can be written as a second-order ordinary differential equation (ODE):

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

where m is the mass of the object, x(t) is the displacement of the mass from its equilibrium position, c is the damping coefficient, k is the spring constant, and F(t) is an external force applied to the system.

To solve this ODE, we first need to specify the initial conditions, which consist of the initial displacement x(0) and initial velocity dx/dt(0) of the mass. We also need to specify the values of the parameters m, c, k, and F(t) if it is not a free vibration problem.
Let's consider a specific example of a spring-mass-damper system with the following parameters:
m = 1 kg; c = 0.2 kg/s; k = 1 N/m; F(t) = 0
We will also assume that the initial displacement of the mass is x(0) = 0.5 m and its initial velocity is dx/dt(0) = 0 m/s.

To solve this ODE, we can use a variety of numerical methods, such as the Euler method, the Runge-Kutta method, or the fourth-order Adams-Bashforth method. Here, we will use the fourth-order Runge-Kutta method, which is a widely used method for solving ODEs.

First, we need to convert the second-order ODE into two first-order ODEs by introducing a new variable y = dx/dt:

$$\frac{dy}{dt} = \left(-\frac{c}{m}\right)y - \left(\frac{k}{m}\right)x\frac{dy}{dt} = y$$

With these two first-order ODEs, we can write the following Python code to solve the ODE using the fourth-order Runge-Kutta method:

Solution-Python code:

```
import numpy as np
import matplotlib.pyplot as plt

# Parameters
m = 1.0  # kg
c = 0.2  # kg/s
k = 1.0  # N/m
F = lambda t: 0.0  # External force

# Initial conditions
x0 = 0.5  # m
v0 = 0.0  # m/s

# Time vector
t0 = 0.0  # s
tf = 10.0  # s
dt = 0.01  # s
t = np.arange(t0, tf+dt, dt)

# Function to evaluate the right-hand side of the ODE
def f(t, y):
    x, v = y
    dxdt = v
    dvdt = (-c/m)*v - (k/m)*x + F(t)/m
    return np.array([dxdt, dvdt])

# Fourth-order Runge-Kutta method
y = np.zeros((len(t), 2))
y[0, 0] = x0
y[0, 1] = v0
for i in range(len(t)-1):
    k1 = dt*f(t[i], y[i])
    k2 = dt*f(t[i] + dt/2, y[i] + k1/2)
    k3 = dt*f(t[i] + dt/2, y[i] + k2/2)
```

Exercise 2. ODEs in heat transfer

Consider a one-dimensional heat transfer problem, where a metal rod of length L is initially at a uniform temperature T0. The left end of the rod is held at a constant temperature T1, while the right end is insulated. The heat transfer within the rod can be modeled by the following one-dimensional heat equation:

$$\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

where T(x, y) is the temperature at position x and time t, ρ is the density of the rod, c_p is its specific heat capacity, and k is its thermal conductivity.

Using the method of separation of variables, find the steady-state solution to this problem and then find the solution to the problem for times t > 0 given the initial condition T(x,0) = T0.

Solution:

Step 1: Find the steady-state solution The steady-state solution is the solution to the heat equation when ∂T/∂t = 0. Thus, we can solve for T(x) using the following differential equation:

$$\frac{\partial^2 T}{\partial x^2} = 0$$

Integrating this twice yields:

T(x) = Ax + B

where A and B are constants of integration. We apply the boundary conditions to determine these constants.

At x = 0, T = T1, so we have:

T(0) = B = T1

At x = L, ∂T/∂x = 0 since the right end is insulated. Thus,

dT/dx = A = 0

Therefore, the steady-state solution is:

T(x) = T1

Step 2: Find the solution for t > 0 given the initial condition T(x,0) = T0 We can assume the solution has the form:

T(x,y) = X(x)T(t)

Substituting this into the heat equation yields:

$$\rho c_p X(x) \frac{dT}{dt} = kT(t) \frac{d^2 X}{dx^2}$$

Dividing both sides by X(x)T(t) and rearranging yields:

$$\frac{1}{kT}\frac{dT}{dt} = \frac{1}{(\rho c_p X(x))}\frac{d^2X}{dx^2}$$

Since the left side of this equation depends only on time and the right side depends only on x, they must be equal to a constant, say -λ2. Thus, we have:

$$\frac{dT}{dt} = -\lambda^2 kT$$

$$\frac{d^2X}{dx^2} = -\lambda^2 \rho c_p X(x)$$

The first ODE has the solution:

$$T(t) = C^{(-\lambda^2 kt)}$$

The second ODE has the general solution:

$$X(x) = D_1 \cos(\lambda x) + D_2 \sin(\lambda x)$$

Applying the boundary conditions, we have:
At x = 0, T = T1, so we have:
X(0)T(t) = T1
D1 = T1
At x = L, ∂T/∂x = 0 since the right end is insulated. Thus,

$$\frac{dX}{dx} = D_1\lambda \cos(\lambda L) + D_2\lambda \sin(\lambda L) = 0$$

Solving for λ yields:
λ = nπ/L, where n is a positive integer
Therefore, the solution for T(x, y) is:

$$T(x,t) = T_1 + \sum\left[C_n^{\left(-\left(\frac{n\pi}{L}\right)^{2kt}\sin\left(\frac{n\pi x}{L}\right)\right)}\right]$$

where the summation is over all positive integers n.

Conclusions:

ODEs are a powerful tool in the field of mechanical engineering, allowing us to model and analyze. Finally, it is concluded that differential equations have had an influence on many areas of life. In addition, they have served, serve and will serve as long as there are problems that can be solved with them. Differential equations are then the fruit of a long process of study, so learning them, understanding them and applying them is to put the result of that study to good use.

References

[1] D. Li, J. Wen, J. Zhang, Recent advances in numerical methods and analysis for nonlinear differential equations, Int. J. Differ. Equations. 2019 (2019). https://doi.org/10.1155/2019/3243510.

[2] H. Chaachoua, A. Saglam, Modelling by differential equations, Teach. Math. Its Appl. Int. J. IMA. 25 (2006) 15–22.

[3] M.L. Abell, J.P. Braselton, Applications of higher-order differential equations, in: Differ. Equations with Math., Elsevier, 2023: pp. 221–282. https://doi.org/10.1016/b978-0-12-824160-8.00010-3.

[4] M.L. Abell, J.P. Braselton, Applications of systems of ordinary differential equations, in: Differ. Equations with Math., Elsevier, 2023: pp. 385–421. https://doi.org/10.1016/b978-0-12-824160-8.00012-7.

[5] J.R. Hass, M.D. Weir, G.B. Thomas, University Calculus: Alternate Edition, CourseSmart eTextbook, (2008) 19. http://math.hawaii.edu/~jamal/tuc01alt_desupps.pdf.

[6] Y. V. Bebikhov, A.S. Semenov, I.A. Yakushev, N.N. Kugusheva, S.N. Pavlova, M.A. Glazun, The application of mathematical simulation for solution of linear algebraic and ordinary differential equations in electrical engineering, IOP Conf. Ser. Mater. Sci. Eng. 643 (2019). https://doi.org/10.1088/1757-899X/643/1/012067.

[7] R.M. May, Simple mathematical models with very complicated dynamics, in: The Theory of Chaotic Attractors, Springer, 2004: pp. 85–93.

[8] Ü. Göktas, D. Kapadia, Methods in mathematica for solving ordinary differential equations, Math. Comput. Appl. 16 (2011) 784–796. https://doi.org/10.3390/mca16040784.

[9] S.R. Moosavi Noori, N. Taghizadeh, Study of Convergence of Reduced Differential Transform Method for Different Classes of Differential Equations, Int. J. Differ. Equations. 2021 (2021). https://doi.org/10.1155/2021/6696414.

[10] J.S. Hwang, Fréchet Differentiability for a Damped Kirchhoff-Type Equation and Its Application to Bilinear Minimax Optimal Control Problems, Int. J. Differ. Equations. 2019 (2019). https://doi.org/10.1155/2019/3238462.

[11] A. Arafa, G. Elmahdy, Application of residual power series method to fractional coupled physical equations arising in fluids flow, Int. J. Differ. Equations. 2018 (2018). https://doi.org/10.1155/2018/7692849.

[12] V. Gill, K. Modi, Y. Singh, Analytic solutions of fractional differential equation associated with RLC electrical circuit, J. Stat. Manag. Syst. 21 (2018) 575–582.

[13] M. Pellicer, J. Sola-Morales, Analysis of a viscoelastic spring–mass model, J. Math. Anal. Appl. 294 (2004) 687–698.

[14] S. Momani, Z.M. Odibat, Fractional green function for linear time-fractional inhomogeneous partial differential equations in fluid mechanics, J. Appl. Math. Comput. 24 (2007) 167–178.

[15] T.M. Atanackovic, B. Stankovic, On a numerical scheme for solving differential equations of fractional order, Mech. Res. Commun. 35 (2008) 429–438.

[16] M.A. Helal, Soliton solution of some nonlinear partial differential equations and its applications in fluid mechanics, Chaos, Solitons & Fractals. 13 (2002) 1917–1929.

[17] K. Nakkeeran, Mathematical description of differential equation solving electrical circuits, J. Circuits, Syst. Comput. 18 (2009) 985–991.

[18] R. Blickhan, The spring-mass model for running and hopping, J. Biomech. 22 (1989) 1217–1227.

[19] R. Weinstock, Spring-mass correction in uniform circular motion, Am. J. Phys. 32 (1964) 370–376.

[20] R. Champion, W.L. Champion, The extension and oscillation of a non-Hooke's law spring, Eur. J. Mech. 26 (2007) 286–297.

[21] G. Kron, Electric circuit models of partial differential equations, Electr. Eng. 67 (1948) 672–684.

[22] H. Yoon, B.D. Youn, H.S. Kim, Kirchhoff plate theory-based electromechanically-coupled analytical model considering inertia and stiffness effects of a surface-bonded piezoelectric patch, Smart Mater. Struct. 25 (2016) 25017.