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BAYESIAN PARAMETER ESTIMATION OF STOCHASTIC DELAY DIFFERENTIAL EQUATIONS IN IRAQI EXCHANGE PARALLEL MARKET

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Abstract:
This paper discussed the Bayesian theory for estimating the parameter of delay in both functions of the drift term and volatility term in the linear stochastic delay differential equations (SDDE). The explicit solution usually is hard to find for this type of stochastic differential equation and Euler-Maruyama method are more likely used in literature reviews as numerical technique to find the parameter estimates. We employed different priors distributions for deriving the posterior distributions with Gibbs sampler algorithm and Metropolis-Hastings algorithm. Also, we applied the Bayesian analysis to discuss the impact of the delay parameter estimates in the parallel currency-exchange market of dollar in the Iraqi. Deviance information criterion (DIC) used to compare the results of SDDE with Geometric Brownian Motion (GBM). The results of Bayesian estimation shows that can SDDE has the less value of DIC which indicates the SDDE have the more ability for studying the behavior of Iraqi parallel currency-exchange market than (GBM).
Key words: stochastic delay differential equation, Geometric Brownian Motion, Bayesian theory, Gibbs sampler algorithm.

1 Introduction

Stochastic differential delay equations are considered one of the most important types of Stochastic differential equations due to the flexibility that provides to deal with the natural framework of the movement of price changes (prices stock or asset prices), that is, it has a way to describe the dynamics of money markets through model this dynamic [3],[5],[13]. In addition, the stochastic differential delay equations are taken considering all the available (historical) information about the price movements in the money market at modeled its behavior. As shown above, this type of equation usually consists of two terms, the first being what is called the deterministic part and the second part is the stochastic part [17],[23]. The first part usually refers to the expected market movement that can be predicted and that it is a risk-free investment because it is not subject to any fluctuations, but the second part, which is the random part, usually models the random changes that occur in the price market because of some unexpected external effects or causes [24],[28],[26]. Moreover, the impact of the delay parameter in the functions of the stochastic delay differential equation has the ability for understanding the behavior of this kind of stochastic differential equations. The investors needs more information about the history of the underlying process, so more information about the delay parameters provides a great tool to make the right decision in investments. Bayesian theory has the natural ability to utilize the expert knowledge (history) and combined it with data information in order to give the accurate estimate of the interested parameter

[14],[19]. As a result, instead of assuming that exchange prices or asset prices are characterized by a Markov property or that they follow the geometric Brownian motion, modeling using stochastic differential delay equations assumes that the development (movement or change) of future stock prices does not depend on the current price of the stock, but rather on previous (history) prices. Motivated strongly by the above discussion, in this paper, we will study the Bayesian estimation for the SDDE parameters. Moreover, we present Gibbs sampler and Metropolis-Hastings algorithms as MCMC techniques to find the estimates of the SDDE parameters. In the remaining sections of this paper, further needed basic concepts and related background are presented in sections 2 and 3. Finally, real data is given to illustrate the behavior of the exchange prices in parallel market b using the SDDE and GBM linear processes.

2 Preliminary and Basic Definitions

The mathematical equation that consists of functions with their derivatives is called differential equation. In real world applications, these functions commonly are representing physical quantities, and their derivatives corresponding to the rates of changes. The differential equations can help in formulating the relationship between these function and its derivatives. Throughout this paper, let (Ω, F, P) be a complete probability space with the filtration$\{F\}_{t\geq 0}$. We consider the following stochastic differential equation as a differential equation that consists of two terms [22],[7],[6]:

$$dx(t) = \mu\big(t, x(t)\big)dt + \sigma\big(t, x(t)\big)dW(t); \quad 0 \leq t \leq T \ldots (1)$$

The initial value of this SDE is the random variable, $x(0) = x_0$ and

$$\mu: [0, T] \times R^n \longrightarrow R^n$$
$$\sigma: [0, T] \times R^n \longrightarrow R^{n+m}$$

Where $\mu(.)$ is the drift term and $\sigma(.)$ Is the volatility term, and W(t) is a standard Brownian motion. Because of Brownian motion, the differential equation is called stochastic differential equation (SDE). The above equation can be represented by using integrals as follows [],[]:

$$X(t+s) = X(t) + \int_t^{t+s} \mu(X(u), u)du + \int_t^{t+s} \sigma(X(u), u)dw(u).$$

If we set $t = 0$, the equation can rewritten as follows,

$$X(t) = X(0) + \int_0^t f(X(u), u)du + \int_0^t g(X(u), u)dw(u) \ldots \ (2)$$

Equation indicates that the stochastic process $\{X_t\}_{t\geq 0}$ is called $It\hat{o}$ process which includes two integrals, the first one is $Riemann$ integral and the second one is called stochastic integral. From the definition of the SDE, it is clear that the historical data (memory or delay) does not considering in the analysis of the finance movements, which is assumed as a drawback of the SDE. Because of that the needed to the SDDE have raised instead of SDE in the analysis of finance markets. The following are very useful and helpful definitions which are the basic concepts for understanding the stochastic differential delay equations. Since we are interest in discuss the delay equation, we will rewrite the SDE in (1) by imposing the delay parameter [1].[2],10],[25].

Def. 1: Let (Ω, F, P)be a complete probability space with filtration$\{F\}_{t\geq 0}$ and let W(t) be a standard Brownian motion on that probability space. The SDDE is defines as follows [27].[16],[18]:

$$dS(t) = f\big(S(t), S(t-\tau)\big)dt + g(S(t), S(t-\tau), t)dW(t)\,; t \geq 0 \quad .....(2)$$
$$S(t) = \phi(t), -\tau \leq t \leq 0$$
$$\text{Where } \emptyset(t) = [-\tau, 0] \longrightarrow R^n$$

Also, we can rewrite the equation (2) in terms of $It\hat{o}$ formula as follows:

$$S(t) = \emptyset(0) + \int_0^t f\big(u, S(u-\tau), S(u)\big)du + \int_0^t g\big(u, S(u-\tau), S(u)\big)dW(u)$$

In this paper, we will follow the [Zhen, 2015] and consider the SDDE have four parameters as defined by,

$$dS(t) = \big(a.S(t) + b.S(t-\tau)\big)dt + \big(c.S(t) + dS(t-\tau)\big)dW(t);$$
$$0 \leq t \leq T$$
$$S(t) = \emptyset(t);\ -\tau \leq t \leq 0 \qquad ...(3)$$

Where $a \text{ and } b$ are the drift parameters, $c \text{ and } d$ are the volatility parameters, and τ is the delay parameters. In context of this paper we are considering a comparison study between the parameter estimates of SDDE and the parameter estimates of the so called Geometric Brownian Motion (GBM) [12],[20],[21]. Next definition is about the GBM.

Def. 2: Geometric Brownian Motion (GBM)

Suppose that S(t) denote the stock price at time $t \geq 0$ which changes randomly. The dynamics of the price of the stock is given as:

$$\frac{ds(t)}{dt} = \mu dt + \sigma w(t)$$

$\frac{ds(t)}{dt}$ is the relative change of price, $\mu > 0$ is the drift term, σ corresponds to diffusion term and W(t) is a standard Brownian motion [11],[15],[4].

The next section will discuss the problem of SDDE parameter estimation with Bayesian theory.

3 Bayesian Estimation of SDDE Parameters

This section aims to estimate the parameters in the SDDE b using the Bayesian theorem and compare it with the traditional model (GBM). In traditional methods usually the estimation of the SDDE parameters required the transition densities of the volatility process which difficult to achieve. This motivates the researchers to follow the Bayesian methods. The Bayesian methods do need small-sample approximations and the inference will be accurate. Moreover, stationarity is not needed [8],[9]. The Bayesian rule depends on the following form,[21]

$$g(\theta|data) = \frac{f(\theta) \times g(\theta)}{\int f(\theta) \times g(\theta) d\theta} \qquad ...(4)$$

or

$$g(\theta|data) \propto f(data|\theta) \times \; g(\theta).$$

The MCMC techniques are the most widely used in the Bayesian parameter estimates which employ the Gibbs sampler algorithm and Metropolis-Hastings algorithm. We will focus on the Bayesian parameter estimates of the SDDE in (2). In Bayesian method, we set $\{S(t)\}_{t\geq 0}$ as the stochastic process with parameter space $\theta = \{a, b, c, d\}$ and under the following assumptions:

1- Suppose that we have N observations of data, y:

$$y = \{S_o, S_1, \dots, S_N\}$$

2- Sampling :

$$g(\theta|y) \propto f(y|\theta) \times g(\theta)$$

$$\propto f(S_o|\theta) \prod_{i=0}^{N-1} f(S_{i+1}|S_i, \theta) \times \; g(\theta).$$

Where, $S_{n+1} = S_n + \left(\alpha . S_n + b . S_{n-N_\tau}\right) . h + \left(C . S_n + d . S_{n-N_\tau}\right) . \Delta W_{n+1}$

and $\Delta W_i \sim N(0, h) \; ; i = 1, \dots, N$

So, the initial solution is $(S_{-N_\tau}, S_{-N_\tau+1}, \dots S_{-1}, S_0)$

Net we will discuss the full conditional distributions.

3.1 The Full Conditional posterior distribution

The key idea in finding the proper posterior distribution is to use a proper prior distribution. We will follow [28],[29] in order to find the posterior distributions of the SDDE parameters (a, b, c, d). The posterior distributions are defined as follows:

1- The full conditional posterior distribution of the parameter a : assume that the likelihood function is

$$f(\tilde{y}_0, \theta) = \prod_{i=1}^{N} \left(\frac{1}{2\pi\sigma_i^2}\right)^{\frac{1}{2}} exp\; exp \left\{-\frac{(S_i - \mu_i)^2}{2\sigma_i^2}\right\} \dots (5)$$

With mean and variance,

$$\mu_i = (ah + 1)S_{i-1} + bhS_{i-N_\tau-1} \; ,$$

$$\sigma_i^2 = c^2(S_{i-1} + dS_{i-N_\tau-1})^2 h$$

Also, suppose that the prior distribution of a is,

$$g(a; 0, \sigma_a^2) = \frac{1}{\sqrt{2\pi\sigma_a^2}} exp\; exp \left\{\frac{-a^2}{2\sigma_a^2}\right\} \dots (6)$$

Then, the posterior distribution of a is,

$$g(a\,|\tilde{y}_0, \tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_N, b, c^2, d) = g(a|S_1, S_2, \dots S_n, \tilde{y}_0, b, c^2, d)$$

$$\propto \prod_{i=1}^{N} f(S_i|\sigma_i^2) . \; g(a|\sigma_a^2)$$

$$\propto \prod_{i=1}^{N} [\frac{1}{\sqrt{2\pi\sigma_i^2}} exp\, exp \left\{-\frac{(S_i-\mu_i)^2}{2\sigma_i^2}\right\}] . \frac{1}{\sqrt{2\pi\sigma_a^2}}$$

$$\propto e^{-\sum_{i=1}^{N} \frac{(S_i-\mu_i)^2}{2\sigma_i^2}} e^{-\frac{a^2}{2\sigma_a^2}}$$

By following some algebraic steps we get the following posterior distribution:

$$\propto exp\left\{-\frac{a^2}{2\left[\sum_{i=1}^{N} \frac{(hS_{i-1})^2}{\sigma_i^2}+\frac{1}{\sigma_a^2}\right]^{-1}}+\sum_{i=1}^{N} \frac{ahS_{i-1}(S_i-S_{i-1}-bhS_{i-N_\tau-1})}{\sigma_i^2}\right\} \dots (7)$$

This is the normal distribution with the following mean and variance,

$$\frac{\left[\sum_{i=1}^{N} hS_{i-1}(S_i-S_{i-1}-bhS_{i-N_\tau-1})/\sigma_i^2\right]}{\left[\sum_{i=1}^{N} \frac{(hS_{i-1})^2}{\sigma_i^2}+\frac{1}{\sigma_a^2}\right]^{-1}},$$

$$\left[\sum_{i=1}^{N} \frac{(hS_{i-1})^2}{\sigma_i^2}+\frac{1}{\sigma_a^2}\right]^{-1}$$

2- The full conditional posterior distribution of the parameter b :

$$g(b\,|\tilde{y}_0,\tilde{y}_1,\tilde{y}_2,\dots,\tilde{y}_N,a,c^2,d) = g(b|S_1,S_2,\dots S_n,\tilde{y}_0\,,a,c^2,d),$$

$$\propto \prod_{i=1}^{N} f(S_i|\sigma_i^2).\ g(b|\sigma_b^2)$$

$$\propto \prod_{i=1}^{N} [\frac{1}{\sqrt{2\pi\sigma_i^2}} exp\, exp \left\{-\frac{(S_i-\mu_i)^2}{2\sigma_i^2}\right\}] . \frac{1}{\sqrt{2\pi\sigma_b^2}}$$

$$\propto e^{-\sum_{i=1}^{N} \frac{(S_i-\mu_i)^2}{2\sigma_i^2}} e^{-\frac{b^2}{2\sigma_b^2}}$$

$$\propto exp\left\{-\frac{b^2}{2\left[\sum_{i=1}^{N} \frac{(hS_{i-N_\tau-1})^2}{\sigma_i^2}+\frac{1}{\sigma_b^2}\right]^{-1}}+\sum_{i=1}^{N} \frac{bhS_{i-1}(S_i-S_{i-1}-ahS_{i-1})}{\sigma_i^2}\right\} \dots (8)$$

This is again the normal distribution with the following mean and variance,

$$\frac{\left[\sum_{i=1}^{N} hS_{i-N_\tau-1}(S_i-S_{i-1}-ahS_{i-1})/\sigma_i^2\right]}{\left[\sum_{i=1}^{N} \frac{(hS_{i-N_\tau-1})^2}{\sigma_i^2}+\frac{1}{\sigma_b^2}\right]^{-1}},$$

$$\left[\sum_{i=1}^{N} \frac{(hS_{i-N_\tau-1})^2}{\sigma_i^2}+\frac{1}{\sigma_b^2}\right]^{-1}$$

3- The full conditional posterior distribution of the parameter c^2 : The prior distribution of the parameter c is inverse Gamma,

$$g(c^2|\alpha,\beta) = \frac{\beta^\alpha}{\alpha}(c^2)^{-\alpha-1} \quad e^{-\frac{\beta}{c^2}}$$

Then the posterior distribution of c is defined as follows:

$$\pi(c^2\,|\tilde{y}_0,\tilde{y}_1,\dots,\tilde{y}_N,a,b,d) = \pi(c^2|S_1,S_2,\dots S_n,\tilde{y}_0\,,a,b,d)$$

$$\propto \prod_{i=1}^{N} \quad f(S_i|\sigma_i^2).\,g(c^2|\alpha,\beta)$$

$$\propto \prod_{i=1}^{N} \quad \frac{1}{\sqrt{2\pi\sigma_i^2}}\, exp\; exp\left\{-\frac{(S_i-\mu_i)^2}{2\sigma_i^2}\right\}.\ \frac{\beta^\alpha}{\alpha}(c^2)^{-\alpha-1} \quad e^{-\frac{\beta}{c^2}}$$

$$\propto \left(\frac{1}{\sqrt{c^2}}\right)^N exp\,\{-\sum_{i=1}^{N} \quad \frac{\left[S_i-\left((ah+1)S_{i-1}-bhS_{i-N_\tau-1}\right)\right]^2}{2hc^2(S_{i-1}+dS_{i-N_\tau-1})^2} * (c^2)^{-\alpha-1} \quad e^{-\frac{\beta}{c^2}}$$

$$\propto (c^2)^{-\left(\frac{N}{2}+\alpha\right)-1} \quad exp\left\{-\frac{(\beta+z)}{c^2}\right\} \qquad \dots(9)$$

Where,

$$z = \sum_{i=1}^{N} \quad \frac{\left[S_i-\left((ah+1)S_{i-1}-bhS_{i-N_\tau-1}\right)\right]^2}{2hc^2(S_{i-1}+dS_{i-N_\tau-1})^2}$$

From (9) we can state that the distribution of the c^2 is inverse gamma with the following parameters:

$$\frac{N}{2}+\alpha = shape\ parameter$$

and

$$\beta + \sum_{i=1}^{N} \quad z_i = scale\ parameter$$

So, e can use the Gibbs sampling algorithm for calculating the mean of the parameters distributions in (7), (8), and (9).

4- Sampling the parameter d: here we will use the Metropolis-Hastings by proposing a certain distribution, $g_i(d^*|d_i)$. The following steps are the basic for calculating the posterior parameter estimates:

a- Set an initial value for $a_0, b_0, {c^2}_0, d_0$ in the step (i+1).

b- Sampling:

$$a_{i+1} \sim g(a_{i+1}|\tilde{y}_0\,,\tilde{y}_1,\dots,\tilde{y}_N,c_i^2,d_i)$$

Sampling: $b_{i+1} \sim g(b_{i+1}|\tilde{y}_0\,,\tilde{y}_1,\dots,\tilde{y}_N,a_{i+1},c_i^2,d_i)$

Sampling: $C_{i+1}^2 \sim g(c_{i+1}^2|\tilde{y}_0\,,\tilde{y}_1,\dots,\tilde{y}_N,a_{i+1},b_{i+1},d_i)$

c- Sampling d^* from $g_i(d^*|d_i)\sim N(d_i, \sigma_d^2)$ and calculate:

$$R = \frac{g(d^*|S_1, S_2, \dots, S_N, \tilde{y}_0, a_{i+1}, b_{i+1}, c_{i+1}^2)/gi(d^*|d_i)}{g(d_i|S_1, S_2, \dots, S_N, \tilde{y}_0, a_{i+1}, b_{i+1}, c_{i+1}^2)/gi(d_i|d^*)}.$$

d- Accept d^* when the first iteration (i++1) is d_{i+1} with probability $min(R, 1)$, otherwise let $d_{i+1} = d_i$

e- Go to step (b) and do (i+2).

3. Real Data Analysis

This section dealt with a practical application of stochastic differential delay equation by analyzing the real data represented in the parallel dollar exchange rates data. We focus on the Bayesian estimation of four parameters of SDDE by using the forms this equation (7), (8), and (9) for estimating the parameters (a, b, and c) through Gibbs sampler algorithm, but the parameter (d) will be estimated by Metropolis algorithm. Also, we will focus on an estimate of two parameters for the Geometric Brownian motion model that referred to it in definition (2) after making (b = 0 and d = 0) in the equation (3). A sample of the daily parallel exchange rates officially published by the Ministry of Planning for the period 2021-2022 was relied upon. Parallel exchange rates are governed by non-random factors and influences that can be predicted and random factors and influences that cannot be predicted. Therefore, traders at the parallel exchange rate rely on their experiences and then make the decision regarding buying or selling. This means that the future decisions of traders at the exchange rate for future sales are not affected by the current price of the currency rather it is affected by their accumulated experience and knowledge of the sudden fluctuations that occur in the exchange rate. It was assumed that the exchange rate of the parallel dollar is subject to a random model adopted by traders in the market based on their experiences, and this random model in which the exchange rate in the market has a delay parameter of one month. We assumed that the delay is presenting in both the average limit of the growth rate of exchange rates and the part of fluctuations of exchange rates. The following graphic shows the movement of the parallel exchange rate of the dollar for the period 2004-2021. Therefore, traders can refer to the pattern of behavior of these prices and identify the repeatable behavior or the rapid changes that occur in the exchange rate. Where exchange rate movement graphics enable traders to know the historical data of prices, and thus provide them with evidence about the level of the exchange rate at which the parallel exchange rate market is shifting.

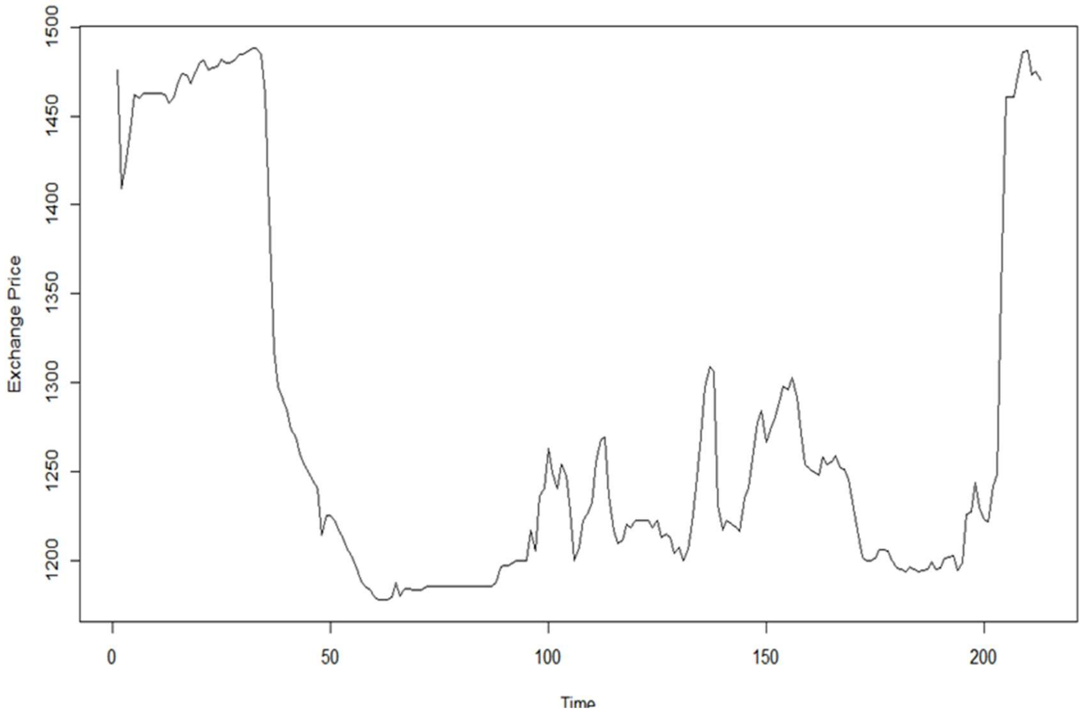

Figure 1: The parallel exchange prices 2004-2021

We notice from Figure (1) that there is a movement in the exchange rates of the dollar that indicates sudden fluctuations during some periods of time and instability in most of the periods. Thus, the delay parameter was assumed to be 3 months, believing that sudden changes occur after 3 months of exchange rates ranging between 1200-1300 Iraqi dinars. Accordingly, we are interested in analyzing this type of financial data, as well as we believe that the prior (historical) information provided by this data will affect the current price movement, which will contribute to making the appropriate decision regarding future trading. The table below shows the estimated parameters and their standard errors using the Bayesian method for both SDDE and GBM models, and assesses the performance of the two models using the DIC criterion.

Table 1: Parameters estimates and DIC values

Models	a	b	c	d	DIC
SDDE (s.e.)	-2.36 (0.712)	1.36 (0.557)	1.39 (0.044)	-1.91 (0.043)	28.03
GBM (s.e.)	-0.44 (0.049)	0 (0)	0.51 (0.0051)	0 (0)	251.74

From Table (1), and for SDDE, we find that the estimated value of parameter a is equal to (-2.36), which means that the average rate of growth of exchange rates is decreasing, and the estimated value of parameter b is equal to (1.36), which means that the average rate of growth of exchange rates with presence of the delay parameter is increasing, which indicates that there is a positive effect of the delay as historical information (priority) on the growth of exchange rates. While we find that the estimated value of parameter c is equal to (1.39), which represents the parameter of deviations or fluctuations (random changes) in exchange rates, as it had a positive effect on the movement of exchange rates without the intervention of any prior or historical information. The estimated value of parameter d is equal to (-1.91), which means that the fluctuations in the movement of exchange rates in the presence of delay effect had a negative impact on the trading movement. Also, from the above table, we find that the value of the DIC criterion for the SDDE model was lower than it is in the BGM model, which indicates that the SDDE model has a preference in representing the movement of dollar exchange rates in the parallel market.

Conclusions

Stochastic models, especially SDDE is among the most common used random models in many sciences fields, especially financial data (exchange rates or stock prices, etc.). The flexibility of the functions of SDDE with the parameter of delay makes this type of random models tractable mathematical model for most of the real world phenomena, such as the financial data. Where the presence of the delay parameter contributes to understanding the movement of financial data as it exploits historical data in drawing a clear picture of the movements of financial data. In addition, we analyze the data of the exchange rates of dollar in the parallel market in Iraq to understand the behavior of exchange rate movements by employing SDDE with the presence of the delay parameter, as well; we used the GBM model and estimating the parameters of this model. The results showed the superiority of the SDDE model in the study of price movements of exchange rate comparing with the geometric Brownian movement model because, which is because of the presence of delay feature that exploit the historical data of exchange rate movements.

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